

When Fewer Bidders Raise Revenue: Bidder Consolidation in Uniform-Price Auctions

Jinsoo Bae

Korea Institute of Finance

`jsbae@kif.re.kr`

September 2026

Abstract

I show that bidder consolidation can raise seller revenue in a multi-unit uniform-price auction with one large bidder and a unit-demand fringe. Consolidating the fringe into equal-capacity blocks replaces its truthful bids with shaded ones, yet those same lower bids make the large bidder less likely to set the price with its own marginal bid, disciplining its pre-existing demand reduction. The net effect is ambiguous in general, but in a benchmark with one rejected bid I identify configurations in which consolidation raises revenue for every distribution within a given regularity class, on a scale from Myerson regularity to a monotone hazard rate. Within that class, the pre- to post-consolidation revenue ratio obeys a sharp upper bound determined only by the number and common capacity of post-consolidation bidders. A bound below one guarantees a revenue gain throughout the class.

Keywords: Bidder consolidation, joint bidding, multi-unit auctions, uniform-price auctions, demand reduction

JEL Classification: D44, D47, L41

1 Introduction

Bidder consolidation is ordinarily expected to reduce seller revenue by removing independent competitors (Markham, 1970; Hendricks and Porter, 1992). In a multi-unit uniform-price auction, however, the initial market may already be distorted by demand reduction: a large bidder shades its marginal bid because that bid can determine payments on units it wins (Engelbrecht-Wiggans and Kahn, 1998; Ausubel et al., 2014). Consolidation can therefore alter an existing strategic distortion as well as the number of competitors.

This paper studies the interaction between these two forces. Consider a large bidder facing a truthful unit-demand fringe. Consolidating the fringe into multi-unit bidders replaces truthful bids with strategically shaded ones. Holding the large bidder’s behavior fixed, this lowers expected revenue: the *price-setter-destruction effect*. But the same lower bids also make any given marginal bid by the large bidder less likely to set the price on its other units. This change can weaken the incentive to suppress that bid. I call this offsetting force the *discipline effect*. Consolidation thus introduces demand reduction among the fringe while potentially alleviating the demand reduction already present.

The discipline effect can be strong enough to raise seller revenue despite the loss of independent competitors. In a simple example with three units and independent uniform values, reorganizing bidder capacities from $(2, 1, 1)$ to $(2, 2)$ raises expected revenue from 0.15 to 0.20. Both remaining bidders shade, yet the seller earns one-third more.

To examine the forces behind this revenue gain, I study a benchmark with one rejected bid. There are $K = Nq - 1$ units, initially contested by one q -unit bidder and $(N - 1)q$ unit-demand bidders. The fringe then consolidates into $N - 1$ blocks of capacity q , leaving N equal-capacity bidders. Unit values are independently drawn from a common distribution F . The single rejected bid determines the uniform price. In this setting, the symmetric increasing post-consolidation equilibrium is efficient and revenue equivalent to

Vickrey, providing a tractable benchmark for the comparison (Engelbrecht-Wiggans and Kahn, 1998, 2002). Within this benchmark, two questions arise: how broadly does the discipline effect operate, and when is it strong enough to outweigh the destruction of truthful bidding?

The first main result shows that discipline is a systematic feature of this environment rather than a peculiarity of the uniform example (Theorem 1). Holding the fringe’s truthful bids fixed, replacing the large bidder’s pre-consolidation strategy with its post-consolidation strategy strictly raises expected revenue for every value distribution on a regularity scale running from Myerson regularity to a monotone hazard rate. Crucially, this is not simply a consequence of the large bidder bidding more at every value after consolidation. At sufficiently low values, the large bidder does bid more: types that previously bid zero on the marginal unit submit positive bids after consolidation. At higher values, however, the response can go the other way. Even under a monotone hazard rate, lower fringe bids can make it attractive for the large bidder to compete for the marginal unit with a lower bid of its own. These possible bid changes do not carry equal weight in the revenue comparison. Because the lowest bid sets the price, increases in low bids are particularly consequential, whereas reductions in higher bids may leave the price unchanged. Theorem 1 establishes that, throughout the regularity scale, the upward responses generate enough additional revenue to outweigh any losses from the downward responses.

A positive discipline effect, however, does not settle the overall revenue comparison, because consolidation also replaces truthful fringe bids with shaded ones. Indeed, an exponential example shows that revenue can fall even when the large bidder raises its bid at every positive value. The second main result identifies when discipline necessarily outweighs this price-setter-destruction effect (Theorem 2). I derive a sharp upper bound on the pre- to post-consolidation revenue ratio, determined only by the number and common capacity of post-consolidation bidders and the regularity class. A bound below one guarantees that

consolidation raises seller revenue for every distribution in the class. Generalized Pareto distributions attain the bound; when it exceeds one, both gains and losses from consolidation are possible.

Intuitively, the bound combines two quantities. The first is the relative revenue advantage that fully truthful bidding would give the pre-consolidation market over the post-consolidation equilibrium. With one rejected bid, both revenues are proportional to the same expected lowest unit value: in the fully truthful pre-consolidation counterfactual, that value is charged on $Nq - 1$ units, whereas revenue equivalence makes post-consolidation expected revenue equal to $N - 1$ Vickrey payments based on that value. This truthful-bidding leverage therefore depends only on N and q . The second quantity is the maximum fraction of this fully truthful revenue that can survive the large bidder's pre-existing demand reduction. The capacity component of the large bidder's price-setting exposure, again fixed by N and q , and the regularity class jointly bound this fraction. Their product gives the largest pre- to post-consolidation revenue ratio possible within the class. If it is below one, consolidation raises seller revenue even for the distribution in the class that is most favorable to pre-consolidation revenue.

Section 2 reviews the related literature. Section 3 presents the motivating example, and Section 4 develops the model. Section 5 separates the two revenue effects and establishes aggregate discipline. Section 6 derives the exact revenue comparison, the sharp class-wide bound, and its capacity implications. Section 7 examines welfare and consolidation incentives, and Section 8 discusses examples beyond the benchmark. Proofs and extensions are collected in the appendices.

2 Related Literature

This paper relates to the literature on bidder coalitions and mergers. Combining independent bids can weaken competition and lower seller revenue (Graham and Marshall, 1987; Waehrer, 1999; Dalkir et al., 2000). The literature has also identified potential benefits of joint bidding through the relaxation of financing constraints, risk sharing, and information pooling (Hendricks and Porter, 1992; Millsaps and Ott, 1985; DeBrock and Smith, 1983; Krishna and Morgan, 1997). This paper examines a mechanism arising from the reorganization of demand within the auction. Consolidation transforms a market with one large bidder and a unit-demand fringe into one with equal-capacity bidders, changing the large bidder's incentive to reduce demand. The potential revenue gain arises without savings in entry costs, relief from financing constraints, or benefits from risk sharing or improved information.

Levin (2004) studies joint bidding in common-value, multi-unit uniform-price auctions, where information pooling interacts with demand reduction. He shows that the revenue benefits of information pooling in single-unit auctions need not extend to a multi-unit setting, and that demand reduction by joint bidders further limits these benefits. In the present paper, demand reduction is already present before consolidation: a large bidder shades its marginal bid while the unit-demand fringe bids truthfully. Joint bidding by the fringe creates additional demand reduction but also changes the bidding incentives of the large bidder, potentially alleviating its existing demand reduction. This response provides an offsetting force to the revenue loss from fringe shading.

The interaction between a large bidder and a truthful fringe is also central to Baisa and Burkett (2018). They study a divisible-good auction in which a privately informed large bidder faces small bidders, and compare uniform-price and discriminatory formats for a given market structure. The present paper holds the uniform-price format fixed and examines the

transition from a large bidder facing a unit-demand fringe to competition among equal-capacity bidders.

The analysis of post-consolidation equilibrium in this paper builds on Engelbrecht-Wiggans and Kahn (1998, 2002), who establish efficiency and revenue equivalence to Vickrey in the symmetric increasing equilibrium when almost every bid wins. Their analysis provides the revenue benchmark for the consolidated market. Comparing this benchmark with the asymmetric market before consolidation connects the paper to the literature on demand reduction in multi-unit auctions, including Ausubel et al. (2014), through the effect of bidder organization on the incentives to shade marginal bids. Elskamp and Kirkegaard (2019) study the symmetric case in which total demand exceeds supply by one unit and show that, for a fixed distribution of the marginal value, the equilibrium marginal bid is the same whenever the number of bidders equals capacity.

The use of revenue bounds also relates to Coey et al. (2019), who use order-statistic gaps to bound the expected revenue loss from counterfactual bidder mergers in single-object ascending auctions. Here, the bound concerns the interaction between consolidation and pre-existing demand reduction in a multi-unit auction, using restrictions on the value distribution and bidder capacities.

The emphasis on how market power is distributed across participants, beyond their number, connects the paper to experimental work on market structure. Holt et al. (1986) show in oral double auctions that reallocating units across traders can alter market power while holding the number of traders fixed. Davis and Holt (1994) distinguish the effects of market power from those of seller numbers in posted-offer markets: reallocating capacity to create market power raises prices, whereas mergers that reduce seller numbers without changing market power have little additional effect on prices. The present paper examines a related issue in a uniform-price auction, where the distribution of demand across bidders determines their exposure to setting the price on inframarginal units.

3 A Motivating Example

A seller offers three identical units to one two-unit bidder and two unit-demand bidders. Every unit value is an independent uniform draw on $[0, 1]$. The three highest bids win, and the single rejected bid determines the price paid on each winning unit. Consolidating the two unit-demand bidders changes capacities from $(2, 1, 1)$ to $(2, 2)$ without changing values or supply.

Before consolidation, the unit-demand fringe bids truthfully. Let X be the large bidder's lower value and Y the lower fringe value. Each is the minimum of two independent uniform draws, so both have cdf $1 - (1 - v)^2$. The large bidder is guaranteed one unit. Raising its marginal bid can win the second unit, but if the bid still loses, it raises the price paid on the guaranteed unit. Balancing these margins gives $v = b + (1 - b)/2$ and the optimal bid

$$b^I(v) = \max\{0, 2v - 1\}. \quad (1)$$

The large bidder therefore gives up the marginal unit whenever $v \leq 1/2$. The shading wedge $(1 - b)/2$ equals the guaranteed quantity divided by the hazard $2/(1 - b)$ of the lowest fringe bid. This ratio measures the large bidder's *price-setting exposure*. Since the price is $\min\{b^I(X), Y\}$, expected revenue is $R^I = 3/20$.

After consolidation, both bidders shade: the symmetric increasing equilibrium marginal bid is $\beta(v) = v^2$ (see also Noussair, 1995). Expected revenue becomes $R^J = 3 \mathbb{E}[\min\{\beta(X), \beta(Y)\}] = 1/5$. The seller gains $1/20$ despite losing an independent bidder.

Two larger effects lie behind this gain. First, holding the fringe's bids truthful, replace $b^I(X)$ with $\beta(X)$. Revenue rises from $R^I = 3/20$ to the hybrid revenue $R^H = 47/140$, an increase of $13/70$: the large bidder bids more at every positive value. Second, replace the truthful fringe bid Y with the consolidated bid $\beta(Y)$. Revenue falls from $R^H = 47/140$ to $R^J = 1/5$, a decrease of $19/140$. The net change is $13/70 - 19/140 = 1/20$. The first change

disciplines pre-existing demand reduction; the second destroys truthful fringe price-setting. The example establishes that discipline can dominate. The analysis below asks whether discipline remains positive more generally and when it is strong enough to deliver a net revenue gain.

4 Model and Equilibrium

4.1 Environment

A seller offers $K = Nq - 1$ identical units, where $N \geq 2$ and $q \geq 2$ are integers. The K highest sealed bids win, and each winner pays the highest rejected bid per unit. There are Nq unit values, drawn i.i.d. from a cdf F with continuous positive density on $[0, \omega)$, $\omega \leq \infty$. Write f for the density of F and $\lambda_F = f/(1 - F)$ for its hazard rate. Bidders are risk neutral. A bidder winning k units obtains the sum of its k highest unit values minus its payment, and each consolidated block maximizes its members' joint payoff.

Before consolidation (regime I , for individual bidding), one bidder controls q units of demand and faces $(N - 1)q$ truthful unit-demand bidders. *After consolidation* (regime J , for joint bidding), the fringe is organized into $N - 1$ blocks of capacity q , leaving N equal-capacity bidders. Throughout, the *large bidder* denotes the bidder that controls q units before consolidation; the new blocks are the *consolidated fringe bidders*. Both regimes contain the same unit values and hold participation, supply, and the pricing rule fixed. Consolidation occurs before values are realized, and each block observes and controls its assigned values. Voluntary formation and internal reporting are outside the model.

Because exactly one bid is rejected, each multi-unit bidder receives at least $q - 1$ units. Its strategic choice reduces to its lowest bid, which alone determines whether it receives $q - 1$

or q units and whether its own bid sets the price. Let

$$G_q = 1 - (1 - F)^q, \quad S = 1 - G_q = (1 - F)^q, \quad \lambda = \frac{g_q}{S} = q\lambda_F. \quad (2)$$

Here g_q denotes the density of G_q . Write $G \equiv G_q$ within a fixed market. The large bidder's lowest value is $X \sim G$. The prospective fringe-block minima are $m_1, \dots, m_{N-1}$, and their minimum $Y = \min_j m_j$ has survival function S^{N-1} , independently of X . Assume finite benchmark revenue:

$$0 < \int_0^\omega [1 - F(p)]^{Nq} dp < \infty. \quad (3)$$

This condition concerns the lowest of all unit values, not the mean of an individual value; a finite mean is not required. The distinction allows the sharp bounds below to cover heavy-tailed regular distributions.

The induced distribution G_q records how capacity changes the relevant marginal value. When q varies, holding the unit-value distribution F fixed does not hold G_q fixed. For a given market, define

$$d \equiv \frac{q-1}{N-1}, \quad L \equiv \frac{Nq-1}{N-1} = q + d. \quad (4)$$

The parameter d captures the capacity component of price-setting exposure; $d/\lambda(p)$ also depends on the distribution. The factor L will link the fully truthful revenue benchmark to post-consolidation revenue.

4.2 Bidding incentives and revenues

Let $A_G = 1 - S^{N-1}$ be the cdf of Y , with density a_G . If the large bidder's marginal bid b beats Y , it wins q units at price Y ; otherwise it wins $q-1$ units at price b . Omitting the

value of those guaranteed units, its expected payoff is

$$\Pi(b; v) = \int_0^b (v - qz) a_G(z) dz - (q - 1)bS(b)^{N-1}. \quad (5)$$

Write $b^I(v)$ for any measurable selection of global maximizers. With $\varphi(p) \equiv p + d/\lambda(p)$, the derivative of (5) with respect to b is

$$\frac{\partial \Pi(b; v)}{\partial b} = a_G(b) [v - \varphi(b)], \quad (6)$$

so the interior first-order condition is $v = \varphi(b)$. Thus, $\varphi(p)$ is the marginal value for which bid p satisfies the first-order condition, generalizing $v = b + (1 - b)/2$ in the example. Raising the marginal bid trades off the chance of winning another unit against a higher price on the $q - 1$ guaranteed units. The wedge $d/\lambda(b)$ measures this exposure. An interior solution need not be a global optimum, so the analysis works with global best responses.

Proposition 1 (Equilibrium and revenues). *In the benchmark:*

- (i) *Truthful bidding is weakly dominant for the fringe. The large bidder has a marginal best response with $0 \leq b^I(v) < v$ for $v > 0$, unique for almost every type.*
- (ii) *Pre-consolidation expected revenue is*

$$R_{N,q}^I = K \int_0^\omega S(p)^{N-1} \mathbb{P}(b^I(X) > p) dp. \quad (7)$$

- (iii) *In the symmetric increasing equilibrium after consolidation, marginal units are allocated efficiently and expected revenue equals Vickrey revenue:*

$$R_{N,q}^J = (N - 1) \int_0^\omega S(p)^N dp. \quad (8)$$

Proof. See Appendix A.1.

Part (iii) uses the almost-every-bid-wins result of Engelbrecht-Wiggans and Kahn (1998, 2002). I select this symmetric increasing equilibrium after consolidation: each block bids its top values truthfully and uses a marginal strategy β with $0 < \beta(v) < v$ for $v > 0$.¹ Since G has a density, choices among exceptional tied best responses before consolidation do not change expected revenue.

The tractability of the benchmark comes from one rejected bid and matched block-minimum distributions. These features reduce bidding to a scalar marginal decision and make the consolidated equilibrium efficient.

5 The Two Revenue Effects

5.1 Separating discipline from the loss of truthful price-setting

Consolidation changes both sides of the price comparison: the large bidder switches from b^I to β , while the lowest fringe bid falls from Y to $\beta(Y)$. A hybrid profile isolates the first change by holding the fringe's bids truthful:

$$R^I = K \mathbb{E}[\min\{b^I(X), Y\}], \quad R^H = K \mathbb{E}[\min\{\beta(X), Y\}], \quad R^J = K \mathbb{E}[\min\{\beta(X), \beta(Y)\}].$$

The hybrid profile need not be an equilibrium. It holds the fringe's bids fixed while measuring the revenue effect of the large bidder's response.

Define the discipline and destruction terms by

$$R^J - R^I = \underbrace{(R^H - R^I)}_{\text{discipline } D} + \underbrace{(R^J - R^H)}_{\text{destruction } T}. \quad (9)$$

¹An all-zero marginal-bid profile cannot be an equilibrium: a bidder with marginal value $v > 0$ can bid $0 < \varepsilon < v$, securing its marginal unit at price zero rather than risking rejection. Asymmetric equilibria can nevertheless exist.

The destruction term is unambiguously negative: $\beta(Y) < Y$, so replacing truthful fringe bids with shaded bids lowers the price with positive probability. Discipline requires more work. Lower fringe bids change both the large bidder's chance of winning the marginal unit and its exposure to pricing the guaranteed units. A higher bid at every value would suffice, but that is not how the general result is obtained.

5.2 Discipline in the aggregate

The large bidder responds to lower rival bids after consolidation, but its optimal response need not be upward at every value. Rather than impose a pointwise ordering, I compare the revenue generated by the two strategies across the standard regularity scale.²

Definition 1 (α -strong regularity). For $\alpha \in [0, 1]$, a cdf H with hazard λ_H is α -strongly regular if its virtual value $\psi_H(v) = v - 1/\lambda_H(v)$ satisfies $\psi_H(y) - \psi_H(x) \geq \alpha(y - x)$ for all $y > x$. Myerson regularity is $\alpha = 0$; MHR is $\alpha = 1$.

Theorem 1 (Aggregate discipline). *Let the unit-value cdf F be α -strongly regular for some $\alpha \in [0, 1]$ and satisfy (3). Then $D > 0$ for every measurable selection of pre-consolidation best responses. No pointwise ordering of b^I and β is required.*

Proof. See Appendix A.3.

The theorem extends the positive discipline effect beyond the uniform example. Low-value types that previously gave up the marginal unit bid positively after consolidation, while some higher-value types may bid less, even under MHR. These changes do not contribute equally to revenue. Against a truthful fringe, a marginal increase in bid b changes expected revenue at rate $K\mathbb{P}(Y > b)$: low bids are more likely to set the price than high bids.

²For α -strong regularity and its auction applications, see Cole and Roughgarden (2014); Cole and Rao (2017). Schweizer and Szech (2019) study an equivalent quantitative strengthening of Myerson regularity.

This weighting alone does not determine the sign. Regularity also restricts how little the large bidder can shade before consolidation. On the cumulative-hazard scale, positive pre-consolidation bids must lie a minimum distance below their types. The post-consolidation bid, by contrast, is an exponentially weighted average of lower type values, with positive weight on zero. The declining price impact described above makes expected price concave in the bid. Jensen's inequality therefore gives a lower bound on hybrid revenue, while regularity gives an upper bound on pre-consolidation revenue. The proof shows that the former exceeds the latter. Taking the minimum of q unit values strengthens regularity, so even Myerson regularity of F is sufficient. Appendix B.1 gives an MHR example in which some types bid less after consolidation.

5.3 Discipline is not sufficient

A positive discipline effect does not ensure that consolidation raises revenue. In the uniform example of Section 3, discipline outweighs the destruction of truthful fringe bidding. An exponential example reverses this ranking, even though the large bidder again bids more at every positive value.

Keep the same market structure, $N = q = 2$, but let unit values follow an exponential distribution with mean $1/2$, the mean of the uniform example. Here, $\varphi(p) = p + 1/4$ is strictly increasing. Since the payoff derivative has the sign of $v - \varphi(b)$, the global best response is

$$b^I(v) = \max\{0, v - \tfrac{1}{4}\}. \quad (10)$$

The post-consolidation equilibrium marginal bid is

$$\beta(v) = v - \tfrac{1}{4}(1 - e^{-4v}). \quad (11)$$

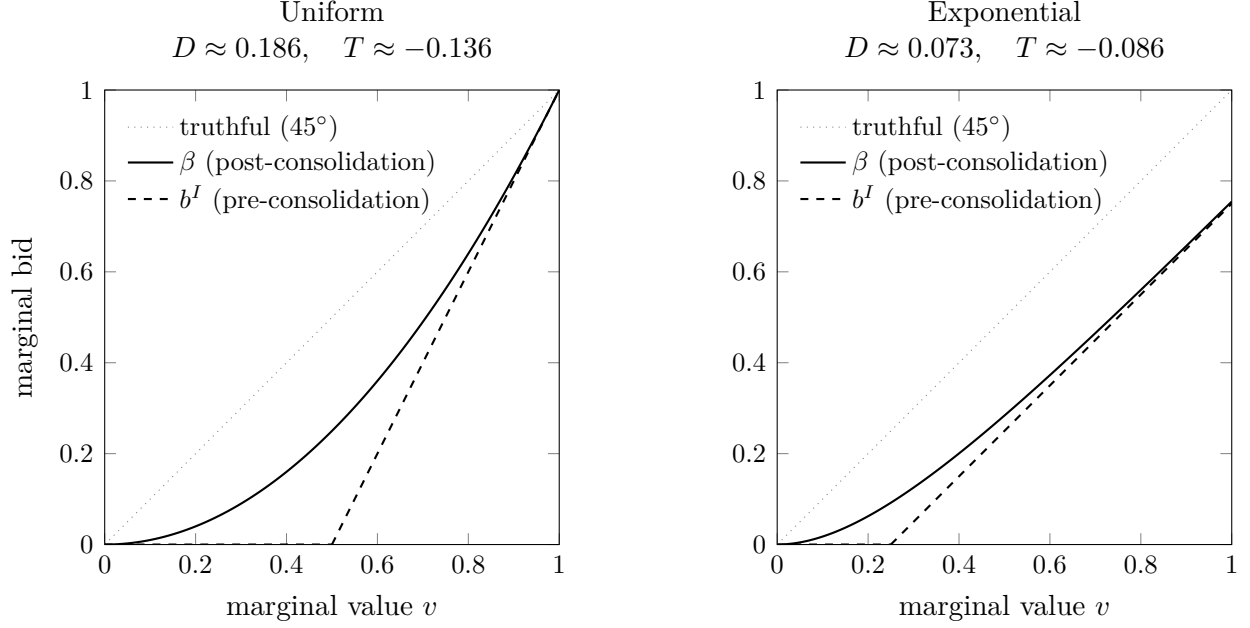


Figure 1: Discipline is not sufficient. Solid and dashed curves show the large bidder's post- and pre-consolidation marginal bids; the dotted line shows truthful bidding. Both panels use $N = q = 2$. The exponential panel displays only $0 \leq v \leq 1$; the reported revenue effects use the full distribution.

For $0 < v < 1/4$, the large bidder switches from a zero bid to a positive bid. For $v \geq 1/4$, its bid increases by $e^{-4v}/4$. Consolidation therefore raises its marginal bid at every positive value, as in the uniform example (Figure 1).

To trace the revenue consequences, retain the hybrid profile of Section 5.1: the large bidder uses β , while the fringe continues to bid truthfully. Expected revenues are

$$R^I = \frac{3}{8e}, \quad R^H = \frac{9 - 3e}{4}, \quad R^J = \frac{1}{8}. \quad (12)$$

The large bidder's response raises revenue by $D = R^H - R^I \approx 0.073$, while replacing the truthful fringe bids with shaded bids lowers it by $T = R^J - R^H \approx -0.086$. Section 6 examines when capacity structure and distributional regularity guarantee a revenue gain.

6 Revenue Comparisons and Sharp Bounds

The decomposition identifies why revenue changes. To determine when the net change must be positive, it is useful to express the same comparison as a revenue ratio. This separates the leverage of truthful bidding from the fraction of that benchmark that survives the large bidder's demand reduction.

6.1 The exact revenue ratio

Suppose first that the large bidder also bids truthfully before consolidation. This profile is not an equilibrium; it serves as a benchmark. Let $R_{N,q}^{I,\text{truthful}}$ denote its expected revenue. The price is then $\min\{X, Y\}$, whose survival function is S^N , and

$$R_{N,q}^{I,\text{truthful}} = K \mathbb{E}[\min\{X, Y\}] = L R_{N,q}^J. \quad (13)$$

The same expected lowest value appears on both sides. In the fully truthful market it is charged on $Nq - 1$ units. In the corresponding Vickrey auction after consolidation, the bidder with the lowest marginal value pays zero, while each of the other $N - 1$ bidders pays that value. Revenue equivalence transfers this total to the uniform-price equilibrium. Consequently, the ratio $L = (Nq - 1)/(N - 1)$ depends only on capacities, although the revenue levels depend on F .

Actual pre-consolidation bidding retains only part of this truthful benchmark. Define, for $0 \leq p < \omega$,

$$\rho(p) \equiv \frac{\mathbb{P}(b^I(X) > p)}{S(p)}, \quad \mu(dp) \equiv \frac{S(p)^N dp}{\int_0^\omega S(t)^N dt}. \quad (14)$$

The numerator of ρ is the probability that the large bidder's actual bid exceeds p ; the denominator is that probability under truthful bidding. Since $b^I(X) \leq X$, the ratio lies in $[0, 1]$. It is also the ratio of actual to fully truthful probabilities that the auction price

exceeds p , because the independent fringe contributes the same factor $S(p)^{N-1}$ to both. The measure μ weights these thresholds by their contribution to benchmark revenue. Write $\bar{\rho} \equiv \int_0^\omega \rho(p) \mu(dp)$ for the weighted average retained fraction.

Proposition 2 (Exact revenue comparison). *For any measurable pre-consolidation best response,*

$$\frac{R_{N,q}^I}{R_{N,q}^J} = L \bar{\rho}. \quad (15)$$

Consolidation raises expected revenue if and only if $\bar{\rho} < 1/L$.

Proof. See Appendix A.4.

Thus $\bar{\rho}$ is the fraction of fully truthful revenue retained after demand reduction. Consolidation raises revenue when this fraction is small enough to offset the truthful market's leverage L .³

The uniform and exponential examples have the same market structure, $N = q = 2$, and hence the same leverage, $L = 3$. Consolidation therefore raises revenue precisely when pre-consolidation demand reduction leaves less than one third of fully truthful revenue. Under uniform values, $\bar{\rho} = 1/4 < 1/3$, giving $R^I/R^J = 3/4$; under exponential values, $\bar{\rho} = 1/e > 1/3$, giving $R^I/R^J = 3/e$. The two examples lie on opposite sides of the same threshold.

6.2 A sharp class-wide bound

Without a restriction on F , no overbidding gives only $R^I/R^J \leq L$. Regularity sharpens the comparison by limiting the fraction of truthful revenue that can survive the large bidder's shading.

³The ratio and the decomposition $D + T$ describe the same comparison through different benchmarks: R^H , in which only the fringe is truthful, and $R^{I,\text{truthful}}$, in which both the large bidder and the fringe bid truthfully. Accordingly, L and $\bar{\rho}$ do not correspond to T and D .

Theorem 2 (Sharp class-wide revenue bound). *Let the unit-value cdf F be α -strongly regular for some $\alpha \in [0, 1]$ and satisfy (3). For any measurable pre-consolidation best response,*

$$\frac{R_{N,q}^I}{R_{N,q}^J} \leq B_{N,q}^F(\alpha) \equiv L \left[1 + (1 - \alpha) \frac{q - 1}{(N - 1)q} \right]^{-q/(1-\alpha)}, \quad (16)$$

where $B_{N,q}^F(1) = Le^{-d}$ is the continuous limit. Generalized Pareto unit values attain the bound, including exponential values at $\alpha = 1$. If $B_{N,q}^F(\alpha) < 1$, consolidation raises revenue for every distribution in the class; if $B_{N,q}^F(\alpha) > 1$, both gains and losses occur within the class.

Proof. See Appendix A.5.

The bound is the largest value of R^I/R^J attainable within the regularity class. Its two factors have distinct roles. The first, L , is the truthful-bidding leverage fixed by capacity. The second is an upper bound on the retained fraction. Because positive pre-consolidation bids must lie a minimum distance below their types on the cumulative-hazard scale, $\rho(p)$ cannot exceed the second factor at any price threshold, and neither can its weighted average $\bar{\rho}$. Generalized Pareto distributions attain the retained-fraction bound at every threshold, making the resulting revenue bound sharp.

A bound above one leaves the sign unresolved rather than predicting a loss. Every regularity class considered here also contains distributions with complete demand reduction, $b^I(X) = 0$ almost surely, for which $R^I = 0$. Both zero and the upper bound are therefore attainable.⁴ With $N = q = 2$ and MHR values the bound is $3/e \approx 1.104$, attained by the exponential example of Section 5.3; the uniform example lies below it at $3/4$.

The distance of the bound from one also has a magnitude interpretation:

$$\frac{R^J - R^I}{R^J} \geq 1 - B_{N,q}^F(\alpha). \quad (17)$$

⁴At $B_{N,q}^F(\alpha) = 1$, consolidation weakly raises revenue throughout the class, and equality is attained.

When the bound is below one, this is a lower bound on the gain as a share of *post-consolidation* revenue. When it is above one, the same inequality limits the possible loss on that scale.

6.3 Capacity comparisons and revenue magnitude

Each before-and-after comparison holds supply and the underlying unit values fixed. Varying N or q , however, compares different markets: total demand and supply $K = Nq - 1$ change with them. The following thresholds describe those market configurations, not successive mergers within a fixed market.

Under MHR, the bound simplifies to

$$\frac{R_{N,q}^I}{R_{N,q}^J} \leq B_{N,q}^F(1) = Le^{-d}, \quad (18)$$

so consolidation raises revenue for every distribution in the MHR class if and only if

$$d > \ln(q + d). \quad (19)$$

Increasing capacity raises the number of units on which a losing marginal bid sets the price, strengthening the incentive for demand reduction before consolidation. Increasing the number of rival blocks works in the opposite direction through $d = (q - 1)/(N - 1)$. In the MHR bound this trade-off appears as the exponential discount e^{-d} against the truthful-bidding leverage $L = q + d$.

For fixed q , the MHR condition eventually fails as N grows; for fixed N , it holds once q is sufficiently large. Proportional growth of q and N is not enough: (19) requires capacity to grow faster than bidder count, on the approximate scale $q \gtrsim (N - 1) \ln q$. The same qualitative conclusions hold at every $\alpha \in [0, 1]$, with weakly larger minimum capacities

Table 1: Minimum capacity q guaranteeing a revenue gain throughout the class

Post-consolidation bidders N	MHR unit values ($\alpha = 1$)	Regular unit values ($\alpha = 0$)
2	3	4
3	5	7
4	9	10
5	12	14

Table 2: Uniform-value examples versus the MHR-class conclusion

(N, q)	R^I/R^J , uniform values	MHR upper bound	MHR-class implication
(2, 2)	0.750	1.104	both signs possible
(2, 3)	0.185	0.677	gain throughout
(3, 2)	1.406	1.516	both signs possible
(3, 4)	0.839	1.227	both signs possible
(3, 5)	0.544	0.947	gain throughout

For uniform unit values, the reported ratios use $R^I/R^J = L(1 - d/q)^q$, obtained from the linear φ in the bidding condition.

required for a revenue guarantee at lower α . These are properties of the class-wide bound, not claims that realized gains are monotone in either parameter for a fixed distribution.

Table 1 gives the smallest integer capacities guaranteeing a revenue gain throughout the MHR and regular classes. Below these capacities, revenue gains remain possible but are not guaranteed. Table 2 shows why a class-wide guarantee differs from a calculation for one distribution. At $(N, q) = (3, 4)$, for example, uniform values produce a gain although other MHR distributions produce a loss. The MHR bound is therefore conservative for this particular distribution, but not loose for the class: some member attains it. Conversely, a bound below one establishes a gain without choosing a particular F .

Ratio improvements should not be confused with large absolute gains. With N and a regular F fixed, raising q increases supply but also lowers the marginal values that determine the price. Under the conditions in Appendix B.2, R^I/R^J tends to zero while both R^J and the absolute gain $R^J - R^I$ tend to zero. The largest absolute gain is therefore reached at a finite capacity.

7 Welfare and Consolidation Incentives

The revenue comparison does not by itself determine who benefits or whether the fringe would choose to consolidate. The common increasing post-consolidation strategy restores efficient allocation of marginal units, and the large bidder benefits from facing shaded fringe bids. The merging fringe need not share those gains.

Proposition 3 (Efficiency and payoffs). *In the benchmark:*

- (i) *realized allocative surplus is weakly higher under consolidation for every value profile. Almost surely, the gain equals $(X - Y)\mathbf{1}\{b^I(X) < Y < X\} \geq 0$;*
- (ii) *the large bidder's interim payoff is weakly higher at every type: its envelope value increases from $\int_0^v A_G(b^I(t)) dt$ to $\int_0^v A_G(t) dt$;*
- (iii) *if pre-consolidation demand reduction is complete, meaning $b^I(X) = 0$ almost surely, each merging fringe block is strictly worse off almost surely.*

Proof. See Appendix A.6.

The allocation improves precisely when pre-existing shading would cause the large bidder to lose its marginal unit to a lower-value fringe unit. Consolidation removes this distortion even when seller revenue falls. The large bidder's payoff also rises because winning the marginal unit against shaded fringe bids is easier than against truthful bids.

These gains do not ensure voluntary consolidation. In the uniform example, the seller gains $1/20$ and the large bidder gains $7/60$, but the merging fringe loses $1/10$ in total; allocative surplus rises by $1/15$. Under complete demand reduction the conflict is particularly stark: fringe units initially win at zero price, whereas after consolidation each block pays a positive price and may lose its marginal unit. In these cases, formation would require transfers, participation benefits, or other motives not captured by a comparison conditional on common bidding control.

8 Discussion

The benchmark isolates the interaction between pre-existing demand reduction and the shading introduced by consolidation. The examples below show that revenue gains remain possible with additional rejected bids and with partial consolidation, while illustrating how the balance between these forces depends on market structure.

With independent uniform unit values on $[0, 1]$ and four units supplied, consolidating two unit-demand bidders changes capacities from $(2, 2, 1, 1)$ to $(2, 2, 2)$, leaving two rejected bids in both markets. In the equilibria constructed in Appendix B.3, expected revenue rises from approximately 0.449 to 0.525. The discipline effect on the two existing multi-unit bidders outweighs the destruction of truthful fringe bidding. The appendix establishes both equilibria and proves the revenue gain using analytical upper and lower bounds; numerical integration gives the reported revenue levels. A contrasting outcome arises with two units supplied: under uniform values, the consolidation $(2, 1, 1) \rightarrow (2, 2)$ admits a post-consolidation split equilibrium that reduces revenue from $3/5$ to zero.⁵

Partial consolidation likewise permits both signs. Under uniform unit values, the change $(3, 1, 1, 1, 1) \rightarrow (3, 3, 1)$, with six units supplied, raises expected revenue from approximately 0.094 to 0.174. By contrast, $(2, 1, 1, 1) \rightarrow (2, 2, 1)$, with four units supplied, lowers revenue from approximately 0.296 to 0.269. Both comparisons preserve one rejected bid and retain a truthful unit-demand bidder. Complete consolidation of the fringe is therefore not necessary for a revenue gain, and retaining some truthful fringe does not ensure one. Appendix B.4 gives the equilibrium constructions and revenue calculations.

The analysis abstracts from three features. The first is a reserve price, which affects both payments and equilibrium selection. Burkett and Woodward (2020) show that reserves

⁵The split equilibrium, in which each bidder bids its higher value on one unit and zero on the other, appears in Engelbrecht-Wiggans and Kahn (1998). Before consolidation, the large bidder bids zero on its second unit, analogous to the complete demand reduction studied by Kagel and Levin (2001); the price is then the lowest of its higher value and the two fringe values, and expected revenue is $3/5$.

can eliminate low-revenue equilibria, in which the price remains at the reserve, when every bidder's opponents have sufficient demand to exceed supply. In the post-consolidation benchmark studied here, each bidder's opponents have total capacity $(N - 1)q < K$, so their equilibrium-elimination result does not apply. The second is unmatched block-minimum distributions, which remove the symmetric Vickrey benchmark. The third is supply uncertainty at the bidding stage, which changes the strategic problem.

The benchmark establishes a positive aggregate discipline effect and a sharp bound identifying when it outweighs fringe shading throughout a regularity class. The extensions show that revenue-enhancing consolidation is not confined to one rejected bid or complete fringe consolidation. Identifying similarly general conditions for these broader environments remains an open question.

Acknowledgements

This paper is based on a chapter of my doctoral dissertation at The Ohio State University. I am grateful to John Kagel, Dan Levin, Jim Peck, Huanxing Yang, Lixin Ye, and Yaron Azrieli for helpful comments. I alone am responsible for any errors. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used ChatGPT (OpenAI) and Claude (Anthropic) in order to improve the language and to assist with derivations and numerical calculations. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

References

- Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2014). Demand reduction and inefficiency in multi-unit auctions. *The Review of Economic Studies* 81(4), 1366–1400.
- Baisa, B. and J. Burkett (2018). Large multi-unit auctions with a large bidder. *Journal of Economic Theory* 174, 1–15.
- Burkett, J. and K. Woodward (2020). Reserve prices eliminate low revenue equilibria in uniform price auctions. *Games and Economic Behavior* 121, 297–306.
- Coey, D., B. Larsen, and K. Sweeney (2019). The bidder exclusion effect. *The RAND Journal of Economics* 50(1), 93–120.
- Cole, R. and S. Rao (2017). Applications of α -strongly regular distributions to Bayesian auctions. *ACM Transactions on Economics and Computation* 5(4), 18:1–18:29.
- Cole, R. and T. Roughgarden (2014). The sample complexity of revenue maximization. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing (STOC)*, pp. 243–252.
- Dalkir, S., J. W. Logan, and R. T. Masson (2000). Mergers in symmetric and asymmetric noncooperative auction markets: the effects on prices and efficiency. *International Journal of Industrial Organization* 18(3), 383–413.
- Davis, D. D. and C. A. Holt (1994). Market power and mergers in laboratory markets with posted prices. *The RAND Journal of Economics* 25(3), 467–487.
- DeBrock, L. M. and J. L. Smith (1983). Joint bidding, information pooling, and the performance of petroleum lease auctions. *The Bell Journal of Economics* 14(2), 395–404.
- Elskamp, R. and R. Kirkegaard (2019). Scale effects in multi-unit auctions. *Games and Economic Behavior* 116, 116–121.
- Engelbrecht-Wiggans, R. and C. M. Kahn (1998). Multi-unit auctions with uniform prices. *Economic Theory* 12(2), 227–258.
- Engelbrecht-Wiggans, R. and C. M. Kahn (2002). Multiunit auctions in which almost every bid wins. *Southern Economic Journal* 68(3), 617–631.
- Graham, D. A. and R. C. Marshall (1987). Collusive bidder behavior at single-object second-price and English auctions. *Journal of Political Economy* 95(6), 1217–1239.
- Hendricks, K. and R. H. Porter (1992). Joint bidding in federal OCS auctions. *The American Economic Review* 82(2), 506–511.

- Hille, E. (1976). *Ordinary Differential Equations in the Complex Domain*. New York: Wiley.
- Holt, C. A., L. Langan, and A. P. Villamil (1986). Market power in oral double auctions. *Economic Inquiry* 24(1), 107–123.
- Kagel, J. H. and D. Levin (2001). Behavior in multi-unit demand auctions: experiments with uniform price and dynamic Vickrey auctions. *Econometrica* 69(2), 413–454.
- Krishna, V. and J. Morgan (1997). (Anti-) Competitive Effects of Joint Bidding and Bidder Restrictions. Working paper, Princeton University, Woodrow Wilson School.
- Levin, D. (2004). The competitiveness of joint bidding in multi-unit uniform-price auctions. *The RAND Journal of Economics* 35(2), 373–385.
- Markham, J. W. (1970). The competitive effects of joint bidding by oil companies for offshore oil leases. In J. W. Markham and G. F. Papanek (Eds.), *Industrial Organization and Economic Development: In Honor of E.S. Mason*. Boston: Houghton Mifflin.
- Millsaps, S. W. and M. Ott (1985). Risk aversion, risk sharing, and joint bidding: a study of outer continental shelf petroleum auctions. *Land Economics* 61(4), 372–386.
- Noussair, C. (1995). Equilibria in a multi-object uniform price sealed bid auction with multi-unit demands. *Economic Theory* 5(2), 337–351.
- Schweizer, N. and N. Szech (2019). Performance bounds for optimal sales mechanisms beyond the monotone hazard rate condition. *Journal of Mathematical Economics* 82, 202–213.
- Waehrer, K. (1999). Asymmetric private values auctions with application to joint bidding and mergers. *International Journal of Industrial Organization* 17(3), 437–452.

A Proofs

Throughout, $X \sim G$ and Y (survival S^{N-1}) are independent; $A_G = 1 - S^{N-1}$ has density $a_G = (N-1)S^{N-2}g_q$; and for independent nonnegative random variables, $\mathbb{E}[\min_i U_i] = \int_0^\infty \prod_i \mathbb{P}(U_i > p) dp$ (layer cake). Note $(q-1)S^{N-1}/a_G = (q-1)/[(N-1)\lambda] = d/\lambda$, which gives the benchmark first-order condition (6).

A.1 Proof of Proposition 1

(i) Truthful bidding is weakly dominant for a unit-demand bidder, whose own bid never sets a price it pays. For the large bidder, differentiating (5) gives

$$\frac{\partial \Pi(b; v)}{\partial b} = a_G(b)[v - \varphi(b)], \quad \frac{(q-1)S(b)^{N-1}}{a_G(b)} = \frac{d}{\lambda(b)}.$$

Since $\varphi(b) = b + d/\lambda(b) > b$, the derivative is negative whenever $b \geq v$, so the payoff is strictly decreasing on $[v, \omega)$. It is continuous on the compact interval $[0, v]$, so a maximum exists and the measurable maximum theorem gives a measurable selection. Every best response therefore satisfies $0 \leq b^I(v) < v$ for $v > 0$.

For almost-everywhere uniqueness, write $\Pi(b; v) = vA_G(b) - C(b)$, where

$$C(b) = q \int_0^b z a_G(z) dz + (q-1)bS(b)^{N-1}.$$

The optimized payoff $U(v) = \sup_{0 \leq b < \omega} \{vA_G(b) - C(b)\}$ is finite, convex, and Lipschitz, because $0 \leq A_G \leq 1$ and $0 \leq U(v) \leq v$. At every differentiability point of U , all optimal bids have the same slope $A_G(b) = U'(v)$. Strict monotonicity of A_G makes the optimal bid unique there. The exceptional types have Lebesgue measure zero and hence G -probability zero.

(ii) The price is $\min\{b^I(X), Y\}$. Independence and layer cake give

$$\mathbb{E}[\min\{b^I(X), Y\}] = \int_0^\omega \mathbb{P}(b^I(X) > p) S(p)^{N-1} dp,$$

which proves (7). No overbidding bounds this integral by $\int S^N$, so revenue is finite.

(iii) Suppose the other $N - 1$ bidders follow a strictly increasing β solving

$$(v - \beta(v))a_G(v) = (q - 1)\beta'(v)S(v)^{N-1}, \quad \beta(0) = 0. \quad (\text{A.1})$$

The solution is

$$\beta(v) = \frac{1}{d} \int_0^v e^{-[H(v)-H(u)]/d} u \lambda(u) du,$$

where $H = -\ln S$. This representation gives $0 < \beta(v) < v$ for $v > 0$, and the ODE then gives $\beta'(v) > 0$. A bidder with marginal value v that bids as type θ obtains, apart from its guaranteed values,

$$\Pi_J(\theta; v) = \int_0^\theta (v - q\beta(z))a_G(z) dz - (q - 1)\beta(\theta)S(\theta)^{N-1}.$$

The ODE gives $\partial \Pi_J(\theta; v)/\partial \theta = (v - \theta)a_G(\theta)$, so reporting $\theta = v$ is globally optimal. Bids above the range of β give the limiting all-win payoff and cannot improve on this maximum. This is the increasing equilibrium characterized by Engelbrecht-Wiggans and Kahn (1998, 2002).

The common increasing strategy rejects the lowest marginal value and therefore implements the Vickrey allocation. The marginal-type payoff is zero at $v = 0$, and the envelope derivative is $A_G(v)$ in both auctions, so interim expected payments coincide. In the Vickrey auction, the block with the lowest marginal value pays zero and each other block pays that value. Expected total payment is $(N - 1)\mathbb{E}[\min\{X, m_1, \dots, m_{N-1}\}]$, giving (8). $\square$

A.2 A common hazard-lag bound

Write $r = 1/\lambda$ and let V be the inverse of H ; thus $H(X) \sim \text{Exp}(1)$ and $V(0) = 0$.

Lemma 1 (Hazard-lag bound). *Suppose G is η -strongly regular, $\eta \in [0, 1]$. Define*

$$h_\eta(d) = \int_0^d \frac{du}{1 + (1 - \eta)u} = \begin{cases} \frac{\ln[1 + (1 - \eta)d]}{1 - \eta}, & \eta < 1, \\ d, & \eta = 1. \end{cases} \quad (\text{A.2})$$

For any measurable selection of pre-consolidation best responses, $t \geq 0$, and $p \in [0, \omega)$,

$$b^I(V(t)) \leq V((t - h_\eta(d))_+), \quad \mathbb{P}(b^I(X) > p) \leq e^{-h_\eta(d)} S(p). \quad (\text{A.3})$$

Proof. By Proposition 1(i), every best response satisfies $b^I(v) < v$ for $v > 0$. Any positive optimum is therefore interior and satisfies $v - b = dr(b)$. Regularity gives $r(b + u) \leq r(b) + (1 - \eta)u$, hence

$$H(v) - H(b) \geq \int_0^{dr(b)} \frac{du}{r(b) + (1 - \eta)u} = h_\eta(d).$$

This proves the first inequality, also trivially valid at a zero bid. The event $b^I(X) > p$ implies $H(X) > H(p) + h_\eta(d)$, giving the second inequality. All hazard integrals remain inside the support. $\square$

For the primitive model, write $r_F = 1/\lambda_F$; then $r = r_F/q$ implies $r(t) - r(z) \leq (1 - \alpha)(t - z)/q$ for $t \geq z$. Thus G is η -strongly regular at $\eta = \alpha_G$, where

$$\alpha_G = 1 - \frac{1 - \alpha}{q}, \quad (\text{A.4})$$

and $\alpha_G > 0$ because $q \geq 2$.

A.3 Proof of Theorem 1

Set $\eta = \alpha_G > 0$ and $h = h_\eta(d)$. The function $\mathcal{A}(b) = \mathbb{E}[\min\{b, Y\}] = \int_0^b S(p)^{N-1} dp$ is increasing and concave, with $\mathcal{A}(0) = 0$ and

$$M \equiv \mathbb{E}[\mathcal{A}(X)] = \int_0^\omega S(p)^N dp = \frac{R_{N,q}^J}{N-1} < \infty.$$

Writing $B(t) = \beta(V(t))$, the equilibrium ODE (A.1) gives $B' = (V - B)/d$, $B(0) = 0$, so

$$B(t) = \frac{1}{d} \int_0^t e^{-(t-s)/d} V(s) ds = \mathbb{E}[V((t - E_d)_+)],$$

where E_d is exponential with mean d . For fixed t , the random variable $V((t - E_d)_+)$ takes values in $[0, V(t)]$. Jensen's inequality and Tonelli's theorem therefore give

$$\mathbb{E}[\mathcal{A}(\beta(X))] \geq \int_0^\infty \int_0^t \mathcal{A}(V(t - e)) \frac{e^{-t-e/d}}{d} de dt = \frac{M}{1+d},$$

where the equality follows by setting $t = s + e$. Lemma 1 and monotonicity of $\mathcal{A}$ give

$$\mathbb{E}[\mathcal{A}(b^I(X))] \leq \int_h^\infty \mathcal{A}(V(t - h)) e^{-t} dt = e^{-h} M.$$

Both expectations are finite because their integrands are bounded by $\mathcal{A}(X)$, whose expectation is $M < \infty$. Combining these inequalities with $D = K \mathbb{E}[\mathcal{A}(\beta(X)) - \mathcal{A}(b^I(X))]$ yields

$$\begin{aligned} D &\geq L \left[\frac{1}{1+d} - e^{-h_\eta(d)} \right] R_{N,q}^J \\ &= \left[\frac{L}{1+d} - B_{N,q}^F(\alpha) \right] R_{N,q}^J > 0, \end{aligned} \tag{A.5}$$

because (A.2) gives $h_\eta(d) > \int_0^d (1+u)^{-1} du = \ln(1+d)$ when $\eta > 0$. □

A.4 Proof of Proposition 2

The minimum $\min\{X, Y\}$ has survival S^N . Truthful pre-consolidation revenue is therefore

$$K \int_0^\omega S(p)^N dp = LR^J,$$

which proves (13). For any measurable best response, substitute (14) into Proposition 1(ii):

$$R^I = K \int_0^\omega S(p)^N \rho(p) dp = LR^J \bar{\rho}.$$

Since $R^J > 0$, comparison with one gives the stated criterion. $\square$

A.5 Proof of Theorem 2

Set $\eta = \alpha_G$. Lemma 1 gives $\rho(p) \leq e^{-h_\eta(d)}$ at every $p < \omega$. Proposition 2 therefore implies

$$\frac{R_{N,q}^I}{R_{N,q}^J} = L \bar{\rho} \leq L e^{-h_\eta(d)}.$$

Substituting $\eta = 1 - (1 - \alpha)/q$ gives (16). More generally, $B_{N,q}^G(\eta) \equiv L e^{-h_\eta(d)}$ is the bound when G is taken directly as an η -strongly regular primitive with $\int S^N < \infty$.

For generalized Pareto unit values with $r_F(v) = \sigma + (1 - \alpha)v$, $\sigma > 0$, the induced reciprocal hazard $r = 1/\lambda$ is affine, so φ is strictly increasing. The payoff derivative therefore changes sign once, and the global best response is zero below $\varphi(0)$ and otherwise solves $\varphi(b) = v$. Hence $\mathbb{P}(b^I(X) > p) = S(\varphi(p))$, and

$$\frac{S(\varphi(p))}{S(p)} = \exp\left(-\int_p^{\varphi(p)} \lambda(t) dt\right) = e^{-h_{\alpha_G}(d)}$$

for every p . Hence $\rho(p) = e^{-h_{\alpha_G}(d)}$, so the bound is attained. Post-consolidation revenue is $R^J = (N - 1)\sigma / (Nq - 1 + \alpha)$, finite and positive. If the bound exceeds one, this distribution

gives a revenue loss. For $F_a(v) = 1 - (1 - v)^a$ on $[0, 1]$ with $0 < a \leq d/q$, we have $\varphi(b) = b + \frac{d}{aq}(1 - b) \geq 1 > v$ for every $b \in [0, 1]$ and every type v . The payoff in (5) is therefore decreasing in b , so $b^I \equiv 0$. This MHR distribution belongs to every regularity class considered here and gives $R^I = 0 < R^J$, establishing the attainable lower endpoint and, when the upper bound exceeds one, the opposite revenue ranking. Equation (17) follows by rearrangement. $\square$

A.6 Proof of Proposition 3

(i) Regime I rejects $\min\{b^I(X), Y\}$, whereas the efficient rejection is $\min\{X, Y\}$. Outside the null set of ties, the allocation is inefficient exactly when $b^I(X) < Y < X$, with surplus loss $X - Y$. The symmetric increasing equilibrium in regime J eliminates this event. With tied values, any rejection among the tied minima is efficient; hence the weak surplus ranking holds for every value profile, while the displayed gain formula holds almost surely.

(ii) In each regime, the large bidder's payoff equals its inframarginal values plus the value of the marginal-unit problem. Its envelope derivative is $A_G(b^I(v))$ before consolidation and $A_G(v)$ afterward. Both value functions are zero at marginal type zero, so integrating and using $b^I(v) \leq v$ gives the stated interim ranking.

(iii) If $b^I(X) = 0$ almost surely, every fringe unit wins at price zero almost surely before consolidation. Consider a prospective fringe block with marginal value m and sum of unit values W . Its pre-consolidation payoff is W . After consolidation, let P denote the realized price. The block either wins all q units, with payoff $W - qP$, or loses its marginal unit, with payoff $W - m - (q - 1)P$. Its realized loss is therefore

$$\begin{cases} qP, & \text{if all } q \text{ units win,} \\ m + (q - 1)P, & \text{if } q - 1 \text{ units win.} \end{cases}$$

All block minima are positive almost surely, and $\beta(v) > 0$ for $v > 0$; hence $P > 0$ almost surely. Both expressions are then strictly positive. $\square$

B Additional Results and Extensions

B.1 An example without pointwise discipline

Let the induced block-minimum hazard on $[0, \infty)$ be

$$\lambda(v) = \lambda_\infty - ae^{-cv}, \quad 0 < a < \lambda_\infty, \quad c > 0,$$

and write $x \equiv cd/\lambda_\infty$. Assume

$$x < 1, \quad acd < (\lambda_\infty - a)^2.$$

The hazard is increasing, so G is MHR. Setting $\lambda_F = \lambda/q$ gives an MHR unit-value distribution whose block minimum has hazard λ .

Under these conditions, $b^I(v) = 0 < \beta(v)$ for $0 < v \leq d/(\lambda_\infty - a)$, whereas $b^I(v) > \beta(v)$ for all sufficiently large v .

Proof. Step 1: before consolidation. Since

$$\varphi'(p) = 1 - \frac{acd e^{-cp}}{\lambda(p)^2} \geq 1 - \frac{acd}{(\lambda_\infty - a)^2} > 0,$$

the function φ is strictly increasing. By (6), the payoff derivative changes sign at most once, so $b^I(v) = 0$ for $v \leq \varphi(0) = d/(\lambda_\infty - a)$ and $\varphi(b^I(v)) = v$ otherwise. Write $s_I(v) = v - b^I(v)$. For $v > \varphi(0)$,

$$s_I(v) - \frac{d}{\lambda_\infty} = \frac{ad e^{-cb^I(v)}}{\lambda(b^I(v))\lambda_\infty} = \frac{ad e^{cs_I(v)}}{\lambda(b^I(v))\lambda_\infty} e^{-cv}.$$

As $v \rightarrow \infty$, $b^I(v) \geq v - d/(\lambda_\infty - a) \rightarrow \infty$, so $\lambda(b^I(v)) \rightarrow \lambda_\infty$ and $s_I(v) \rightarrow d/\lambda_\infty$. Hence

$$e^{cv} \left[s_I(v) - \frac{d}{\lambda_\infty} \right] \rightarrow \frac{ad}{\lambda_\infty^2} e^x.$$

Step 2: after consolidation. The representation of β in Appendix A.1 is

$$\beta(v) = \frac{1}{d} \int_0^v e^{-[H(v)-H(u)]/d} u \lambda(u) du.$$

Integration by parts gives

$$s_J(v) \equiv v - \beta(v) = \int_0^v e^{-[H(v)-H(u)]/d} du.$$

With $w = v - u$,

$$H(v) - H(u) = \lambda_\infty w - \frac{a}{c} e^{-cv} (e^{cw} - 1),$$

and therefore

$$s_J(v) = \int_0^v e^{-\lambda_\infty w/d} \exp[\varepsilon(v)(e^{cw} - 1)] dw, \quad \varepsilon(v) = \frac{a}{cd} e^{-cv}.$$

Step 3: a lower bound. Using $e^y \geq 1 + y$,

$$s_J(v) \geq \frac{d}{\lambda_\infty} (1 - e^{-\lambda_\infty v/d}) + \varepsilon(v) \int_0^v e^{-\lambda_\infty w/d} (e^{cw} - 1) dw.$$

Because $x < 1$, the last integral converges to

$$\frac{d}{\lambda_\infty - cd} - \frac{d}{\lambda_\infty} = \frac{d}{\lambda_\infty} \frac{x}{1 - x}.$$

Also $e^{-\lambda_\infty v/d} = o(e^{-cv})$, so

$$\liminf_{v \rightarrow \infty} e^{cv} \left[s_J(v) - \frac{d}{\lambda_\infty} \right] \geq \frac{ad}{\lambda_\infty^2} \frac{1}{1-x}.$$

Step 4: comparison. For $0 < x < 1$, $e^{-x} > 1 - x$ implies $e^x < 1/(1 - x)$. The preceding limits therefore give $s_J(v) > s_I(v)$, and hence $\beta(v) < b^I(v)$, for all sufficiently large v . For $0 < v \leq \varphi(0)$, $b^I(v) = 0 < \beta(v)$ by Proposition 1. $\square$

The example $N = q = 2$, $\lambda_\infty = 3$, $a = 2$ and $c = 0.475$ satisfies both conditions. Its aggregate discipline effect remains positive by Theorem 1.

B.2 Capacity depth and absolute gains

The bound in Section 6.3 concerns a ratio. The following result separates its limiting behavior from the absolute revenue gain along a sequence with N and F fixed while q and supply $Nq - 1$ grow.

Proposition 4 (Deep capacity). *Fix $N \geq 2$ and an α -strongly regular unit-value distribution F with $F(0) = 0$ and $\int_0^\omega [1 - F(p)]^{2N} dp < \infty$. Suppose F has a density continuous at zero with $0 < f(0) < \infty$. Then, as $q \rightarrow \infty$,*

$$\begin{aligned} R_{N,q}^J &\sim \frac{N-1}{Nf(0)q}, \\ \frac{R_{N,q}^I}{R_{N,q}^J} &\longrightarrow 0, \quad R_{N,q}^J - R_{N,q}^I \sim \frac{N-1}{Nf(0)q} \longrightarrow 0. \end{aligned} \tag{B.1}$$

Proof. Write $\bar{F} = 1 - F$ and $n = Nq$. Continuity of f at zero gives $-\ln \bar{F}(p) = f(0)p + o(p)$. For any $0 < \varepsilon < f(0)$, choose $\delta > 0$ such that

$$(f(0) - \varepsilon)p \leq -\ln \bar{F}(p) \leq (f(0) + \varepsilon)p, \quad 0 \leq p \leq \delta.$$

Integrating the resulting exponential bounds yields

$$\frac{1}{f(0) + \varepsilon} \leq \liminf_{n \rightarrow \infty} n \int_0^\delta \bar{F}(p)^n dp \leq \limsup_{n \rightarrow \infty} n \int_0^\delta \bar{F}(p)^n dp \leq \frac{1}{f(0) - \varepsilon}.$$

For the remaining tail,

$$n \int_\delta^\omega \bar{F}(p)^n dp \leq n \bar{F}(\delta)^{n-2N} \int_\delta^\omega \bar{F}(p)^{2N} dp \longrightarrow 0.$$

Letting $\varepsilon \downarrow 0$ proves

$$n \int_0^\omega \bar{F}(p)^n dp \longrightarrow \frac{1}{f(0)}. \quad (\text{B.2})$$

Equation (8) then gives the asymptotic formula for R^J .

For $\alpha < 1$, the bracket in (16) converges to $1 + (1 - \alpha)/(N - 1) > 1$, its negative exponent is proportional to q , and $L = O(q)$. For $\alpha = 1$, the bound is $L \exp[-(q - 1)/(N - 1)]$. Thus $B_{N,q}^F(\alpha) \rightarrow 0$ in both cases. Theorem 2 implies $R^I/R^J \rightarrow 0$, and hence $R^J - R^I \sim R^J$. $\square$

The gain is eventually positive but converges to zero, so its maximum over integer capacities is reached at a finite q . The maximizing capacity is distribution-specific and need not be the first capacity at which the class-wide bound falls below one.

B.3 Two rejected bids

Six unit values are independent and uniform on $[0, 1]$, and four units are supplied. The price is the highest rejected bid, which is the second-lowest of the six bids. Capacities are $(2, 2, 1, 1)$ before consolidation and $(2, 2, 2)$ after the two unit-demand bidders merge. A two-unit bidder holds the ordered pair (w, v) , $w \geq v$, of two draws. Within this subsection R^I , R^H and R^J denote expected revenue in the first market, in the hybrid profile defined below, and in the second market.

Proposition 5 (Two rejected bids). *Both markets have a Bayesian Nash equilibrium in which unit-demand bidders bid truthfully, each two-unit bidder bids its higher value truthfully, and all two-unit bidders use a common marginal strategy. Before consolidation the marginal bid is zero for $v \leq 1/2$ and strictly increasing for $v > 1/2$; after consolidation it is strictly increasing on $[0, 1]$. In these equilibria*

$$R^I \leq \frac{2187}{4480} < \frac{52}{105} \leq R^J.$$

Proof. Step 1: marginal bids. For a two-unit bidder, let H_1 be the cdf of the second-lowest of its four rival bids, h_1 its density, and P_1 the probability that exactly one rival bid lies below a given level. With the higher bid equal to w , the payoff derivative with respect to the marginal bid $b \leq w$ is

$$U_b(b; v) = (v - b) h_1(b) - P_1(b). \quad (\text{B.3})$$

By the dominance argument in Engelbrecht-Wiggans and Kahn (1998) (Lemma 2.2), it suffices to consider bids (w, b) with $0 \leq b \leq v$, and truthful bidding is optimal for a unit-demand bidder. If an increasing function t satisfies $(t(b) - b)h_1(b) = P_1(b)$, then

$$U_b(b; v) = h_1(b) [v - t(b)], \quad (\text{B.4})$$

which changes sign once, from positive to negative, at the bid with $t(b) = v$. That bid is a global best response.

Step 2: local solutions. The two equilibrium conditions below take the form $x z'(x) = f(x, z)$, with f analytic near the origin, $f(0, 0) = 0$, and $f_z(0, 0)$ not a positive integer. Such an equation has a unique analytic solution with $z(0) = 0$ (Hille, 1976, Theorem 11.1.1, p. 403).

Step 3: after consolidation. Let t denote the inverse of the marginal strategy, $A = (1-t)^2$ and $B = 1 - A - b^2$. A rival block has no bid, one bid, or two bids below b with probabilities A , B and b^2 . With two rival blocks, $H_1 = 1 - A^2 - 2AB$ and $P_1 = 2AB$, and (B.3) at $v = t(b)$ gives

$$\beta'(v) = \frac{2(v - \beta)(2v - v^2 - \beta^2)}{(1 - v)[2v(1 - v) + (v - \beta)^2]}. \quad (\text{B.5})$$

Substituting $\beta = vz$ puts (B.5) in the form of Step 2 with $f_z(0,0) = -1$, so a solution exists near zero, with $\beta(v) = v^2 + \frac{1}{2}v^4 + O(v^5)$. The right-hand side of (B.5) is positive for $0 < \beta < v < 1$, equals $2v/(1-v)$ at $\beta = 0$ and zero at $\beta = v$. The solution therefore stays in $0 < \beta(v) < v$, is strictly increasing, and extends to $[0, 1)$. If its limit at one were $\ell < 1$, the right-hand side would behave as $2(1 + \ell)/(1 - v)$, which is not integrable; hence $\beta(1) = 1$. By (B.4) the strategy is a global best response. Evaluating the right-hand side of (B.5) at $\beta = v^2$ gives $2v + 4v^3/(2 + v - v^2) > 2v$, so the solution cannot cross v^2 from above, and

$$\beta(v) \geq v^2. \quad (\text{B.6})$$

Step 4: before consolidation. A two-unit bidder faces one block and two truthful unit-demand bidders. With A and B as above,

$$H_1 = 1 - (1 - b^2)(1 - b)^2 - 2Ab(1 - b), \quad P_1 = B(1 - b)^2 + 2Ab(1 - b).$$

Here $\partial H_1 / \partial b$ holds t fixed, whereas the density is the total derivative along $t(b)$:

$$h_1(b) = \frac{\partial H_1}{\partial b} + 4(1 - t)b(1 - b)t'(b).$$

Taking right-hand limits at zero, the condition $(t - b)h_1 = P_1$ reads $(2t - 1)[1 - (1 - t)^2] = 0$ and admits the positive-cutoff solution $t(0) = 1/2$. The positive-bid branch is constructed

from this boundary value. For $b > 0$, (B.3) gives

$$t'(b) = \frac{Q(b, t)}{4(1-t)b(1-b)(t-b)}, \quad Q = P_1 - (t-b) \frac{\partial H_1}{\partial b}. \quad (\text{B.7})$$

Set $z(b) = t(b) - 1/2$. Multiplying (B.7) by b gives $bz' = f(b, z)$ with f analytic near $(0, 0)$, $f(0, 0) = 0$ and $f_z(0, 0) = -3/2$, so Step 2 yields a unique analytic local branch with $t(b) = \frac{1}{2} + \frac{13}{14}b^2 + O(b^3)$. Let $y = (1-t)/(1-b)$. Along (B.7), $y' = 5/[4(1-b)] > 0$ at $y = 1/2$ and $y' = -5/(24b) < 0$ at $y = 2/3$, and $1/2 < y < 2/3$ for small $b > 0$. Hence

$$\frac{1+2b}{3} < t(b) < \frac{1+b}{2}, \quad 0 < b < 1. \quad (\text{B.8})$$

The solution has no interior singularity, satisfies $t > b$, and tends to one. Moreover $Q > 0$ near zero, and $\partial Q/\partial b = 2(1-t)^2(1+b-2t) + 12(t-b)b(1-b) > 0$ inside (B.8); at a first zero of Q one would have $t' = 0$ and $dQ/db > 0$, which is impossible. Thus $t' > 0$. Let $b^I(v) = 0$ for $v \leq 1/2$ and $b^I(v) = t^{-1}(v)$ for $v > 1/2$. By (B.4) the payoff is decreasing at every positive bid when $v \leq 1/2$ and has a single peak at $b^I(v)$ when $v > 1/2$. Two marginal bids of zero are both rejected, so ties at zero do not affect the allocation. By (B.8),

$$b^I(v) \leq u(v) \equiv \frac{3v-1}{2} \mathbf{1}\{v > 1/2\}. \quad (\text{B.9})$$

Step 5: revenue. The price is nondecreasing in every bid. Replacing b^I by u therefore bounds R^I from above, and replacing β by v^2 bounds R^J from below. The price exceeds p when at most one of the six bids lies below p . With A the probability that neither bid of a block lies below p and $B = 1 - A - p^2$, the survival functions of the price are $A^2(1-p^2) + 2AB(1-p)^2$ before and $A^3 + 3A^2B$ after consolidation. Under u , $A = 1/4$ for $p < 1/4$ and $A = 4(1-p)^2/9$ otherwise; under v^2 , $A = (1 - \sqrt{p})^2$. Integrating and multiplying by four units gives $R^I \leq 2187/4480$ and $R^J \geq 52/105$. $\square$

In the hybrid profile the two existing blocks use β and the unit-demand bidders bid truthfully. The bound (B.6) gives $R^H \geq 2402/3465$, so discipline satisfies $R^H - R^I \geq 90943/443520 > 0$, while $R^J - R^H < 0$ because the new block replaces a truthful bid v by $\beta(v) < v$. With positive probability, exactly one other bid lies below $\beta(v)$ and all remaining bids exceed v , so the price falls from v to $\beta(v)$. Numerical integration of (B.5) and (B.7) gives $R^I \approx 0.449$, $R^H \approx 0.721$ and $R^J \approx 0.525$.

B.4 Partial consolidation

Consider a one-rejected-bid market with two blocks of capacity q , one truthful unit-demand bidder, and supply $K = 2q$. Unit values are uniform on $[0, 1]$. If both blocks use an increasing marginal strategy β , the lowest rival bid facing a block is the minimum of the rival block's marginal bid and the unit-demand bidder's value. Its hazard λ_{rival} at $b = \beta(v)$ is $q\lambda_F(v)/\beta'(v) + \lambda_F(\beta(v))$, and the first-order condition $v = b + (q - 1)/\lambda_{\text{rival}}(b)$ becomes

$$\beta'(v) = \frac{q \lambda_F(v) [v - \beta(v)]}{(q - 1) - \lambda_F(\beta(v)) [v - \beta(v)]}, \quad \beta(0) = 0. \quad (\text{B.10})$$

With $x = 1 - v$ and $z = (1 - \beta)/x$, equation (B.10) reads

$$\frac{d\beta}{dv} = \frac{q z (z - 1)}{(q - 2) z + 1}, \quad (\text{B.11})$$

Since $\beta = 1 - xz$, $\beta'(v) = z + x dz/dx$, so $x dz/dx = z(2z - q - 1)/[(q - 2)z + 1]$. Separating variables and imposing $z = 1$ at $x = 1$ gives

$$x^3 = \frac{3 - 2z}{z} \quad (q = 2), \quad x = z^{-1/4}(2 - z)^{3/4} \quad (q = 3). \quad (\text{B.12})$$

Proposition 6 (Partial consolidation). *Let F be uniform. Both changes below preserve one rejected bid and a truthful unit-demand bidder.*

(i) With four units, $(2, 1, 1, 1) \rightarrow (2, 2, 1)$ lowers revenue from $8/27$ to $10 - 24 \ln(3/2) \approx 0.269$. The post-consolidation marginal bid is $\beta_P(v) = 1 - 3(1 - v)/[2 + (1 - v)^3]$.

(ii) With six units, $(3, 1, 1, 1, 1) \rightarrow (3, 3, 1)$ raises revenue from $3/32$ to $999/5 - 288 \ln 2 \approx 0.174$.

Proof. A block with marginal value v and marginal bid b has payoff derivative proportional to $v - \varphi_P(b)$, where $\varphi_P(b) = b + (q - 1)/\lambda_{\text{rival}}(b)$. By (B.11), $d\beta/dv > 0$ for $z > 1$, so the strategies in (B.12) are increasing. Along an increasing solution of (B.10), φ_P coincides with the inverse of the strategy and is increasing, so the first-order condition gives the global best response.

(i) Before consolidation the lowest of three fringe values has hazard $3/(1 - b)$, so $\varphi_P(b) = b + (1 - b)/3$, $\mathbb{P}(b^I(X) > p) = [\frac{2}{3}(1 - p)]^2$ and $R^I = 4 \int_0^1 (1 - p)^3 [\frac{2}{3}(1 - p)]^2 dp = 8/27$. After consolidation, (B.12) with $q = 2$ gives $z = 3/(2 + x^3)$, which is β_P . With $u = (1 - v)^3$,

$$R^J = 4 \int_0^1 (1 - v)^4 [1 - \beta_P(v)] \beta'_P(v) dv = 24 \int_0^1 \frac{u(1 - u)}{(2 + u)^3} du = 10 - 24 \ln \frac{3}{2}.$$

(ii) Before consolidation the lowest of four fringe values has hazard $4/(1 - b)$, so $\varphi_P(b) = b + (1 - b)/2$, $b^I(v) = \max\{0, 2v - 1\}$, $\mathbb{P}(b^I(X) > p) = [\frac{1}{2}(1 - p)]^3$ and $R^I = 6 \int_0^1 (1 - p)^4 [\frac{1}{2}(1 - p)]^3 dp = 3/32$. After consolidation, (B.12) with $q = 3$ gives the strategy, with $d\beta/dv = 3z(z - 1)/(z + 1) > 0$ for $1 < z < 2$, $\beta(0) = 0$ and $\beta(1) = 1$. The price exceeds $p = \beta(v)$ with probability $x^6(1 - p)$, and $1 - p = zx = [z(2 - z)]^{3/4}$. Changing variables to $z \in [1, 2)$,

$$R^J = 6 \int_0^1 x^6(1 - p) dp = 9 \int_1^2 \frac{(z - 1)(2 - z)^5}{z} dz = \frac{999}{5} - 288 \ln 2. \quad \square$$